Quaternion Parameters in the Simulation of a Spinning Rigid Body

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This article is included here because, like the foregoing, it is concerned with a programming technique for error reduction in a very important class of problems. It complements well the previous article, for as that one provides a valuable technique in the accurate simulation of the translational motion of a body, this one discusses an important formulation of coordinate transformation which is essential to the accurate simulation of the attitude behavior of a rotating body. Though the mathematics behind the technique is less than simple, the eventual computer program is elegant and the advantage to be gained from the use of the program suggests that it be seriously considered in any simulation of moving bodies where rotation and coordinate transformations are important. JM

ABSTRACT

The use of quaternions in describing the orientation of a rigid body allows all possible attitudes to be simulated. The problem of gimbal lock encountered when using the more commonly understood Euler angles is avoided. The analog implementation of the quaternion description is given together with the transformations between quaternion parameters and Euler angles in order that the latter may be available for display.

INTRODUCTION

In simulations of the motion of a rigid body not considered as a point mass, it is necessary to represent the orientation of the body with respect to some detached reference frame—not only to keep track of its change with time, but also to permit the effect of forces, which are readily defined relative to the body, to be transformed for inclusion in translational equations of motion specifying velocity components and consequent position relative to the reference frame. For this purpose a frame of three orthogonal axes is defined within and fixed to the body, and its rotation with respect to the detached reference frame is simulated in terms of appropriate components of the corresponding spin vector.

For illustration consider the typical computational loop shown in Figure 1 where the interchange between the two frames is readily appreciated. If we begin with the translational velocity vector defined by components (u_o, v_o, w_o) in the reference frame, a transformation to body-fixed components (u_B, v_B, w_B) allows the determination of the angles of attack (α) and sideslip (β), commonly used to determine the attitude of the body with respect to the airstream. With a knowledge of the aerodynamic coefficients, the body-related components of force (X_B, Y_B, Z_B) can be obtained from the angles of attack and sideslip, and the translational velocity. The loop is closed by resolving these body-related components of force to equivalent components in the detached reference frame, so that they may be integrated to produce the components of velocity with which we began.

The orientation of a rigid body is commonly specified by the three nonorthogonal Euler angles of yaw, pitch, and roll (ψ, θ, ϕ), which permit the reference frame to be aligned by successive rotations with the body-fixed frame. These angles are computed by integrating the components (P, Q, R) of the spin vector according to the standard gimbal equations:

$$\begin{aligned} \dot{\psi} &= (R\cos\varphi + Q\sin\phi)/\cos\theta \\ \dot{\theta} &= -R\sin\varphi + Q\cos\varphi \\ \dot{\phi} &= P + \dot{\psi}\sin\theta \end{aligned} \tag{1}$$

The equations themselves forecast a computational difficulty (inherent in any 3-angle specification of orientation) known as gimbal lock, for if the second angle θ approaches ± 90 degrees, the rates of change of angles ψ and ϕ (in this case, yaw and roll) approach infinity. A change in the order of the sequence of rotations helps in some simulation situations where it can be assumed one of the three angles does not vary beyond a range ± 70 degrees. Making that angle the second one of the sequence avoids the difficulty, but, for the general case where this restriction is not permissible, the addition of redundant variables is essential if gimbal lock is not to complicate the simulation.

One well-known formulation[1,2] uses nine direction cosines to replace the three Euler angles, an expensive computational procedure because six additional constraining equations (those of orthonormality) are added. Commonly the set of equations employed has six differential equations with three constraints, plus three algebraic equations for the three remaining direction cosines.

Intuitively it seems that it should be possible to solve the computational difficulty by adding a single redundant variable. Rather than adding one variable to the three Euler angles it is better to define four quaternion parameters. This note discusses this set of parameters, and in particular provides the relationships which permit the Euler angles (which are typically used to display the orientation of the body) to be computed from them. The simplicity of the corresponding analog computer diagrams is also demonstrated.

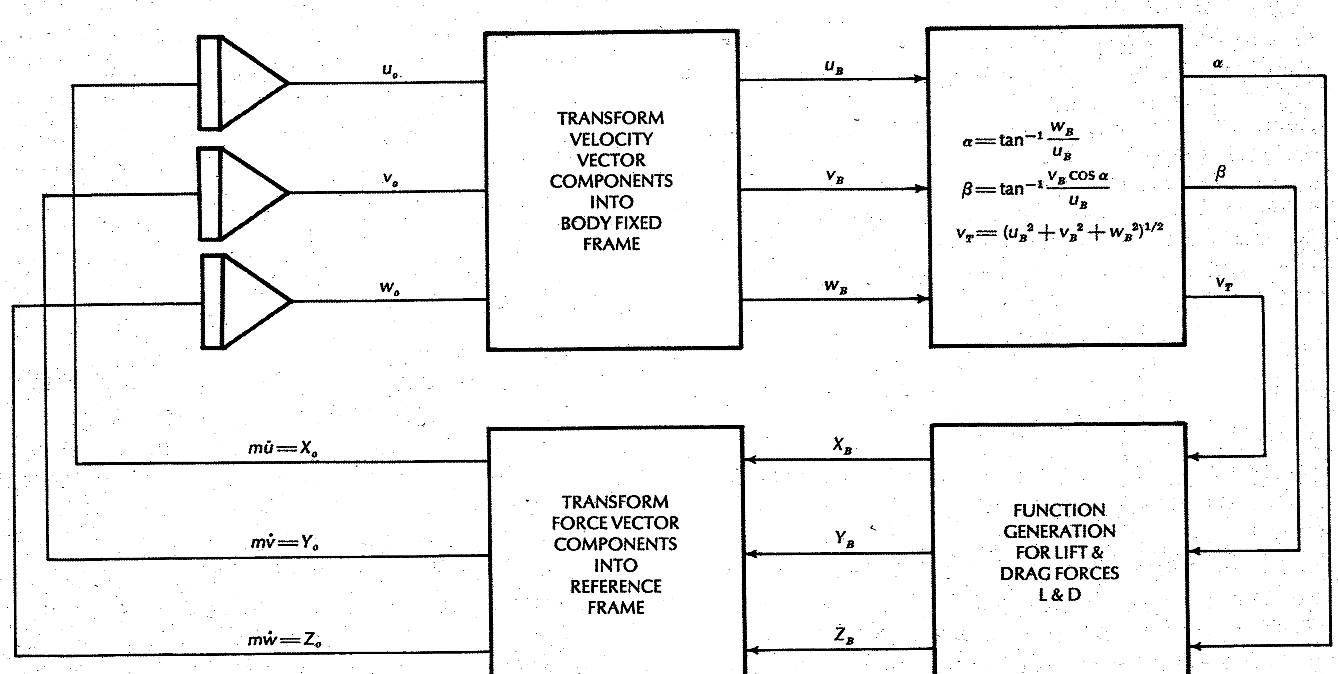

Figure 1—Computational Loop Showing Vector Component Transformation

QUATERNIONS

Euler showed that the orientation of one frame of axes with respect to another is uniquely determined by a single rotation δ about a particular direction $\vec{S}$. The chosen direction $\vec{S}$ can be specified by the three angles of inclination α, β, γ which it makes with the axes of the reference frame, and thus we have four parameters to establish the orientation. The single constraint equation in this four parameter system is:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (2)$$

In place of the commonly-used sequence of Euler angles for moving the reference frame to the body-fixed frame, the procedure employing quaternion parameters can be stated as:

1. Rotate the reference frame, using an orthogonal matrix A of direction cosines, to cause the X-axis to be aligned with the chosen direction $\vec{S}$.
2. Rotate around direction $\vec{S}$ through an angle δ.
3. Rotate through an inverse matrix A^{-1} until the frame is aligned with the body-fixed axes.

Clearly both the direction of $\vec{S}$ and the value of δ must be chosen to cause the alignment — Euler showing that it is always possible to choose appropriate values.

Mathematically the rotation of components (X_o, Y_o, Z_o) of a vector in the reference frame to equivalent components (X_b, Y_b, Z_b) in the body-fixed frame can be written as:

$$\begin{vmatrix} X_b \\ Y_b \\ Z_b \end{vmatrix} = \begin{vmatrix} \cos\alpha & m_1 & n_1 \\ \cos\beta & m_2 & n_2 \\ \cos\gamma & m_3 & n_3 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\delta & \sin\delta \\ 0 & -\sin\delta & \sin\delta \end{vmatrix} \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} X_o \\ Y_o \\ Z_o \end{vmatrix} \qquad (3)$$

The overall transformation must be orthonormal, providing six independent conditions to establish the values of the undefined direction cosines ($m_1, m_2, m_3, n_1, n_2, n_3$).[3,4]

The form of this transformation and the equations for computing the parameters are both simplified by a change of variables. Let

$$\begin{aligned} e_0 &= \cos(\delta/2) \\ e_1 &= \cos\alpha \sin(\delta/2) \\ e_2 &= \cos\beta \sin(\delta/2) \\ e_3 &= \cos\gamma \sin(\delta/2) \end{aligned} \qquad (4)$$

Differential equations for the quaternion parameters can be obtained by considering the components of a unit vector in the two frames when the body is spinning, the spin vector having components (P_b, Q_b, R_b) in the body frame.

They are:[3,4]

$$\begin{aligned} \dot{e}_0 &= -\frac{1}{2}(e_1P_b + e_2Q_b + e_3R_b) \\ \dot{e}_1 &= \frac{1}{2}(e_0P_b + e_2R_b - e_3Q_b) \\ \dot{e}_2 &= \frac{1}{2}(e_0Q_b + e_3P_b - e_1R_b) \\ \dot{e}_3 &= \frac{1}{2}(e_0R_b + e_1Q_b - e_2P_b) \end{aligned} \qquad (5)$$

Note how the equation for e_0 is symmetrical in (e_1 e_2 e_3) and (P_b Q_b R_b), and that the equations for e_1, e_2, and e_3 can be obtained from each other by a rotation of the components (P_b, Q_b, R_b) and (e_1, e_2, e_3).

The constraint equation becomes:

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \qquad (6)$$

An algebraic constraint can be applied to a set of differential equations by adding to each equation a component proportional to the gradient of the square of the constraint, ε say. In this case

$$\varepsilon = 1 - (e_0^2 + e_1^2 + e_2^2 + e_3^2) \qquad (7)$$

Thus in the computer program one mechanises in place of equation (5), the following:

$$\begin{aligned} \dot{e}_0 &= -\frac{1}{2}(e_1P_b + e_2Q_b + e_3R_b) + K\varepsilon e_0 \\ \dot{e}_1 &= \frac{1}{2}(e_0P_b + e_2R_b - e_3Q_b) + K\varepsilon e_1 \\ \dot{e}_2 &= \frac{1}{2}(e_0Q_b + e_3P_b - e_1R_b) + K\varepsilon e_2 \\ \dot{e}_3 &= \frac{1}{2}(e_0Q_b + e_1Q_b - e_2P_b) + K\varepsilon e_3 \end{aligned} \qquad (8)$$

where K is set to a very high value.

Note that the subscripts (0, 1, 2, 3) are carefully chosen and ordered to establish a symmetry in the derived parameters. Subscript 0 is made to correspond to the rotation δ, 1 corresponds to α, 2 to β, and 3 to γ.

This choice is better than those used in previous publications for it leads to equations and relationships that are readily remembered by their symmetry.

With the substitution of these parameters, the total transformation matrix becomes:

$$\begin{vmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1e_2 + e_0e_3) & 2(e_1e_3 - e_0e_2) \\ 2(e_1e_2 - e_0e_3) & e_0^2 + e_2^2 - e_1^2 - e_3^2 & 2(e_2e_3 + e_0e_1) \\ 2(e_1e_3 + e_0e_2) & 2(e_2e_3 - e_0e_1) & e_0^2 + e_3^2 - e_1^2 - e_2^2 \end{vmatrix}$$

The analog computer program which solves the equations is given in Figure 2. The components (P_b, Q_b, R_b) of the spin vector are obtained by a separate integration of applied moments. Although using sixteen (16) multipliers and four (4) squaring cards, it is a comparatively simple circuit. It is interesting to note the "oscillator" loops shown by the heavy lines, the feedback around any integrator being by way of a second integrator and an inverting amplifier. The gain in such a loop is proportional to the square of one of the components of spin. Thus whenever the body is spinning the values of the quaternion parameters change with time proportionally to sin ωt, where ω has an instantaneous value $P_b/2$, $Q_b/2$ or $R_b/2$.

This program has been checked for accuracy in constant rotation by applying values of 1 rad/sec to P_b, Q_b, and R_b for an accumulated angle of 3600 degrees. The final displayed angles were in error by less than 5 degrees, using typical analog components with no attempt made to choose well-aligned multipliers and high-accuracy components.

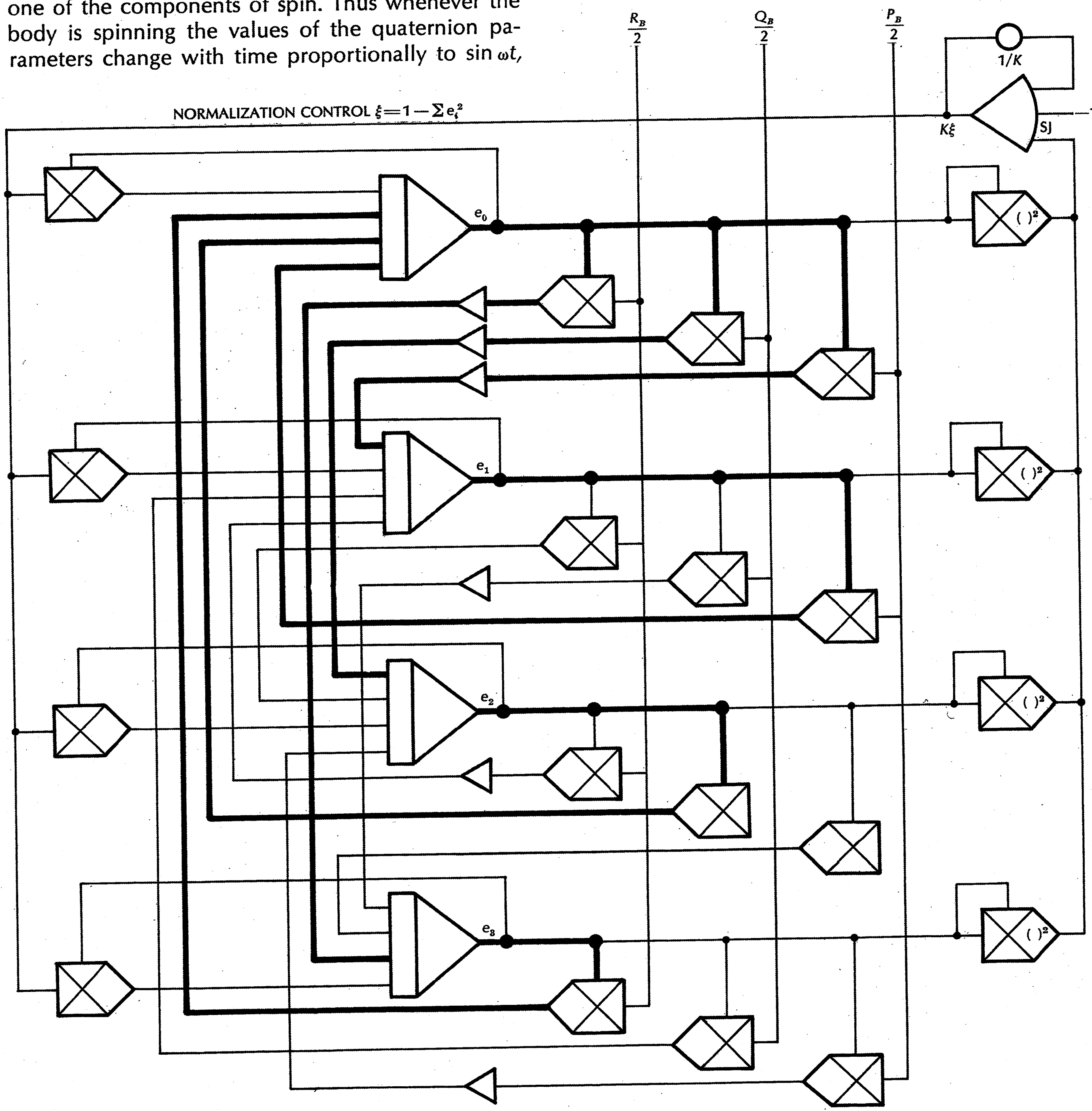

Figure 2 – Integration of Body Rates to Quaternions

QUATERNION-EULER ANGLE TRANSFORMATION

The attitude of the body within the reference frame must be equally determined by the quaternion parameters and the Euler angles, so there must exist an algebraic transformation from one to the other. As there are four quaternion parameters and only three Euler angles, the transformation can be expressed in four different ways, indicating that, to obtain a particular single-valued expression, additional constraints must be stated and implemented on the computer.

If one permits the Euler angles to have all values from $-\pi$ to $+\pi$, there is within them a two-fold ambiguity (two sets of values for ψ, θ, ϕ which represent the same condition). To avoid this ambiguity, it is common to consider the pitch angle, θ, to have values only within the range $-\pi/2 < \theta \leq +\pi/2$. Then as the pitch angle passes through $+\pi/2$, all angles are considered to change by π. For example, as the small increment ε goes through zero,

$$\left.\begin{array}{l} \psi = 2\pi/3 \\ \theta = \pi/2 + \varepsilon \\ \phi = -\pi/4 \end{array}\right\} \equiv \left\{\begin{array}{l} -\pi/3 \\ -\pi/2 + \varepsilon \\ +3\pi/4 \end{array}\right. \qquad (9)$$

These jumps in value present difficulties in any simulation where the values of the angular components are computed for use in transformations of force vectors. Thus, it is common to use the sines and cosines of the angles which are continuous functions, and derive the angles themselves simply for display purposes.

The transformation matrix, expressed earlier in terms of the quaternion parameters, can also be stated in terms of the Euler angles and equivalently direction cosines: *i.e.*

$$\begin{vmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

where:

$$\begin{aligned} \ell_1 &= \cos\psi \cos\theta \\ \ell_2 &= \sin\psi \cos\theta \\ \ell_3 &= -\sin\theta \\ m_1 &= \cos\psi \sin\theta \sin\varphi - \sin\psi \cos\varphi \\ m_2 &= \sin\psi \sin\theta \sin\varphi + \cos\psi \cos\varphi \\ m_3 &= \cos\theta \sin\varphi \\ n_1 &= \cos\psi \sin\theta \cos\varphi + \sin\psi \sin\varphi \\ n_2 &= \sin\psi \sin\theta \cos\varphi - \cos\psi \sin\varphi \\ n_3 &= \cos\theta \cos\varphi \end{aligned} \qquad (10)$$

By equating the terms of these matrices to those in the quaternion transformation given earlier, five relationships between Euler angles and the quaternion parameters may be used to solve for the former. The five equations are:

$$2(e_1e_3 - e_2e_0) = \ell_3 = -\sin\theta \qquad (11a)$$
$$2(e_1e_2 + e_3e_0) = \ell_2 = \sin\psi \cos\theta \qquad (11b)$$
$$e_0^2 + e_1^2 - e_2^2 - e_3^2 = \ell_1 = \cos\psi \cos\theta \qquad (11c)$$
$$2(e_2e_3 + e_1e_0) = m_3 = \sin\phi \cos\theta \qquad (11d)$$
$$e_0^2 - e_1^2 - e_2^2 + e_3^2 = n_3 = \cos\phi \cos\theta \qquad (11e)$$

Provided the pitch angle θ is restricted to the range $-\pi/2 < \theta \leq +\pi/2$, θ is single-valued, and a unique inverse sine value exists — see Figure 3.

$$\therefore \theta = \sin^{-1}[-\ell_3] = \sin^{-1}[2(e_2e_0 - e_1e_3)] \qquad (12)$$

The angle ψ can be obtained from equations (11b) and (11c), the latter giving:

$$\cos\psi = \ell_1/\cos\theta \qquad (13)$$

which can be satisfied by $+\psi$ or $-\psi$. The ambiguity may be removed by considering the sign of $\sin\psi$, which is the same as the sign of ψ (see Figure 4).

Thus equation (13) may be written

$$\psi = \cos^{-1}[\ell_1/\cos\theta] \cdot \mathrm{sgn}[\sin\psi] \qquad (14)$$

where

$$\mathrm{sgn}(x) = +1 \qquad x \geq 0$$

and

$$\mathrm{sgn}(x) = -1 \qquad x < 0$$

From equation (11b)

$$\mathrm{sgn}[\ell_2] = \mathrm{sgn}[\sin\psi] \cdot \mathrm{sgn}[\cos\theta] \qquad (15)$$

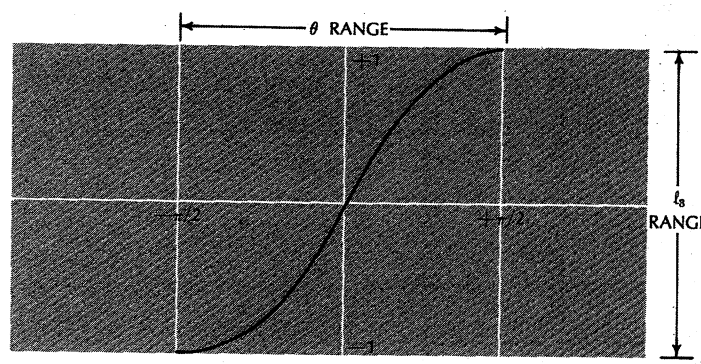

Figure 3 — Transformation from $\ell_3 \rightarrow \theta$

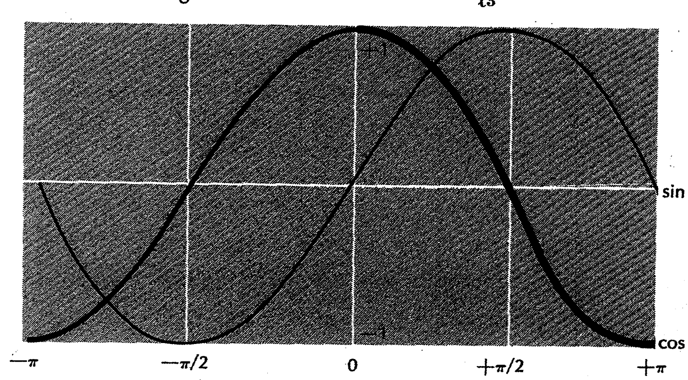

Figure 4 — Relationship between Cosine and Sign of Sine

Because of the restriction on θ, $\cos\theta$ is always positive so that

$$\text{sgn}\,[\ell_2] = \text{sgn}\,[\sin\psi] \tag{16}$$

and hence from equation (14)

$$\psi = \cos^{-1}[\ell_1/\cos\theta] \cdot \text{sgn}\,[\ell_2] \tag{17}$$

or in terms of the quaternion parameters

$$\psi = \cos^{-1}\left[\frac{e_0{}^2 + e_1{}^2 - e_2{}^2 - e_3{}^2}{\cos\theta}\right] \cdot \text{sgn}\,[e_0e_3 + e_1e_2] \tag{18}$$

Similar reasoning leads to the expression for φ

$$\varphi = \cos^{-1}[n_3/\cos\theta] \cdot \text{sgn}\,[m_3] \tag{19}$$

or

$$\varphi = \cos^{-1}\left[\frac{e_0{}^2 + e_3{}^2 - e_1{}^2 - e_2{}^2}{\cos\theta}\right] \cdot \text{sgn}\,[e_0e_1 + e_2e_3] \tag{20}$$

The analog computer program for these equations is shown in Figure 5 and some of the features are worth noting. Only one function generator, that for $\cos\theta$, requires two amplifiers and is indicated by a symbol to represent both the shaping circuits and the two amplifiers combined. By using the transformation:

$$\cos x = -\sin\,[x - (\pi/2)]$$

Equation (17) becomes

$$|\psi| = \sin^{-1}[-\ell_1/\cos\theta] + \pi/2$$

and the inverse sine is limited to

$$-\pi/2 < \sin^{-1}[\ell_1/\cos\theta] \leq \pi/2$$

Similar equations apply to $|\phi|$ and $\sin^{-1}[n_3/\cos\theta]$. As the functions are now monotonic over the restricted range, passive diode shaping networks may be used with a current feed to the summing junction of the single output summing amplifier.

As $\cos\theta$ is used as a divisor, a hard zero limit is applied to prevent it from ever becoming slightly negative. There are occasions when $\cos\theta$ is zero and the resulting quotients are given by 0/0. At this point ψ and ϕ are not individually defined, but $\phi + \psi$ is constant. In typical applications, it is not essential to enforce this constancy for $\cos\theta$ is zero only momentarily as the pitch angle passes through 90 degrees. Even when, as in a missile launch, the pitch angle remains at 90 degrees for many seconds, the con-

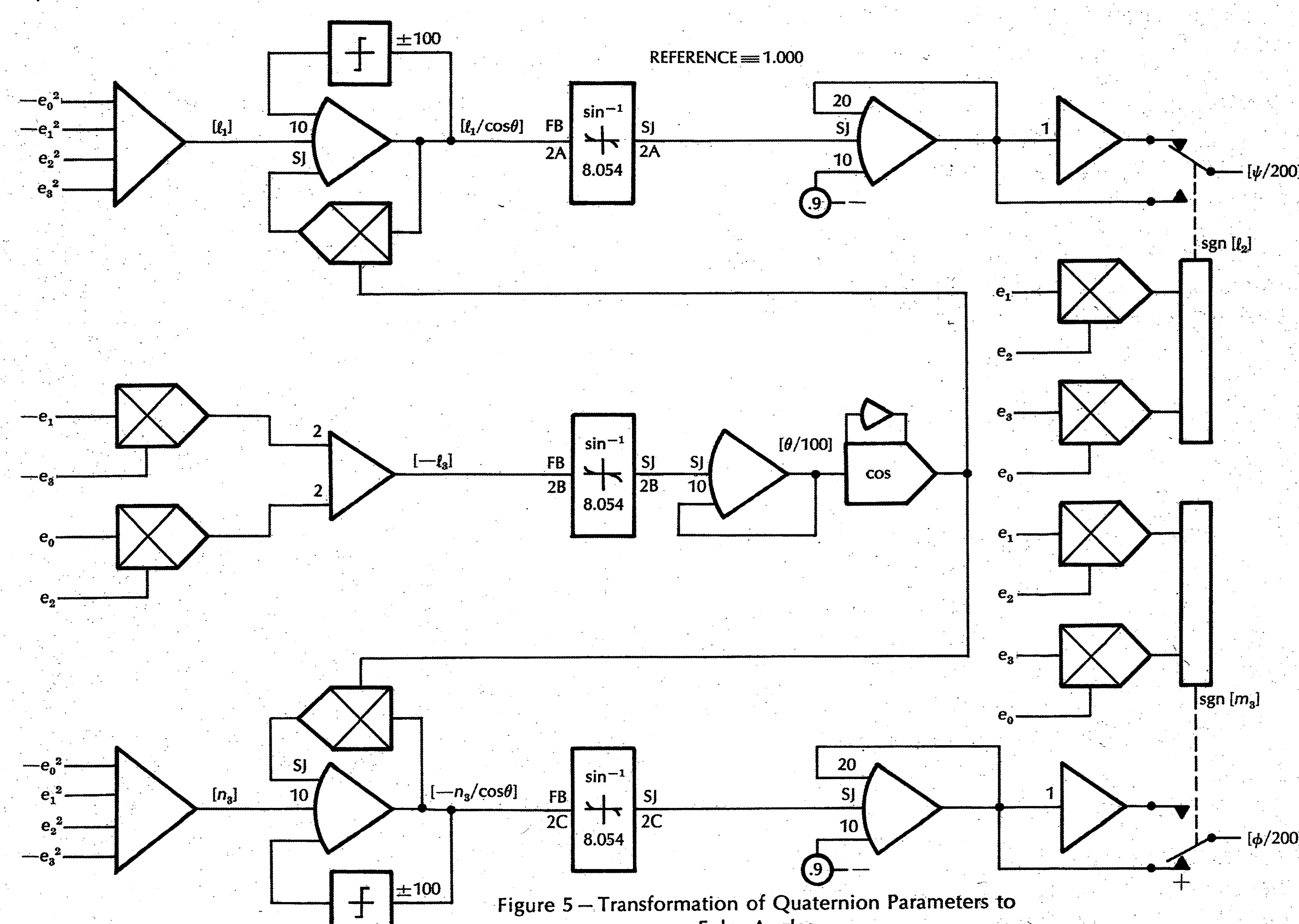

Figure 5 — Transformation of Quaternion Parameters to Euler Angles

stancy is probably not essential for the angles are only generated for observation and recording. Note that when an orientation display for a manned vehicle employs a gyro-driven "8-ball," the drives are from signals representing sine and cosine of the angles rather than the angles and thus the 8-ball does not drift, no matter the time period for which the pitch angle is 90 degrees.

For the application of arbitrarily initial values, it is useful to have expressions for the quaternion parameters in terms of given values of ψ, θ, and ϕ. Manipulation of the expressions for equivalent elements of the transformation matrix gives the following relationships.

$$
\begin{aligned}
e_0 &= \pm\left[\cos\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2}+\sin\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}\right]\\
e_1 &= \pm\left[\cos\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}-\sin\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2}\right]\\
e_2 &= \pm\left[\cos\frac{\psi}{2}\sin\frac{\theta}{2}\cos\frac{\phi}{2}+\sin\frac{\psi}{2}\cos\frac{\theta}{2}\sin\frac{\phi}{2}\right]\\
e_3 &= \pm\left[-\cos\frac{\psi}{2}\sin\frac{\theta}{2}\sin\frac{\phi}{2}+\sin\frac{\psi}{2}\cos\frac{\theta}{2}\cos\frac{\phi}{2}\right]
\end{aligned}
\tag{21}
$$

One may choose to use either the positive or the negative sign, provided the same sign is used for all parameters.

CONCLUSIONS

Quaternions have the advantage of avoiding gimbal lock, allowing simulation of a tumbling vehicle, and are readily implemented with simple analog programs. At the same time, particularly if servo-multipliers are used, the reduction in frequency by a factor of two will reduce phase shift problems compared to a direction-cosine mechanization.

Interpretation of quaternions in terms of body attitude is not simple, but a transformation to Euler angles is practical and requires only a small amount of analog equipment. The accuracy obtained by the use of quaternions in vector transformation is comparable with that available by any other procedure.

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